A kinetic approach to Bose-Einstein condensates: Self-phase modulation and Bogoliubov oscillations

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(Dated: Received 8 November 2004)

Abstract

A kinetic approach to the Bose-Einstein condensates (BECs) is proposed, based on the Wigner-Moyal equation (WME). In the quasi-classical limit, the WME reduces to the particle number conservation equation. Two examples of applications are: i) a self-phase modulation of a BE condensate beam where we show that a part of the beam is decelerated and eventually stops as a result of the gradient of the effective self-potential; ii) the derivation of a kinetic dispersion relation for sound waves in the BECs, including a collisionless Landau damping.

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PACS numbers: 02.70Lq, 03.35.Fi, 32.80.Pj, 67.40.Db

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I. INTRODUCTION

Presently, the Bose-Einstein condensates (BECs) provide one of the most active and creative areas of research in physics [1, 2]. The dynamics of the BECs are usually described by a nonlinear Schrödinger equation (known in this field as the Gross-Pitaevskii equation (GPE) [3, 4]), which determines the evolution of a collective wave function of ultra-cold atoms in the BECs, evolving in the mean field self-potential.

In this paper, we propose the use of an alternative but nearly equivalent approach to the physics of BECs, based on a kinetic equation for the condensate. We also show that this kinetic theory can lead to a more complete understanding of the physical processes occurring in the BECs, not only by providing an alternative method for describing the system, but also by improving our global view of the physical phenomena. It is our hope that this will also lead to the discovery of new aspects of BECs.

The key point of our present approach is the use of a Wigner Moyal equation (WME) for the BECs, describing the spatio-temporal evolution of the appropriate Wigner function [5]. Wigner functions for the BECs were discussed in the past [6, 7] and the WME has been sporadically used [8]. But no systematic application of the WME to BECs has previously been considered. In the quasi-classical limit, this equation reduces to the particle number conservation equation, which is a kinetic equation formally analogous to the Liouville equation, but with a nonlinear potential. A description of the BECs in terms of the kinetic equation is adequate to deal with a series of problems, as exemplified here, and can be seen as an intermediate (in accuracy) between the GPE and the hydrodynamic equations usually found in the literature.

The manuscript is organized in the following fashion. In Section 2, we establish the WME and discuss its approximate version as a kinetic equation for the Wigner function. We then apply the kinetic equation to two distinct physical problems. The first one, considered in Section 3, is the self-phase modulation of a BEC beam. A similar problem has been studied numerically in the past [9]. Here, we derive explicit analytical results and show that a part of the BEC beam is decelerated and eventually comes to a complete halt as a result of the collective forces acting on the condensate. The second example is considered in Section 4, where we establish a kinetic dispersion relation for sound waves in the BECs, giving a kinetic correction to the usual Bogoliubov sound speed [12, 13] and predicting the occurrence of

Landau damping [10, 11]. Our description of Landau damping is significantly different from that previously considered for transverse oscillations of BECs [14]. Finally, in Section 5, the virtues and limitations of the present kinetic approach are briefly discussed.

II. WIGNER MOYAL EQUATION FOR THE BOSE CONDENSATE

It is known that for an ultra-cold atomic ensemble, and in particular for BECs, the ground state atomic quantum field can be replaced by a macroscopic atomic wave function ψ. In a large variety of situations, the evolution of ψ is determined by the GPE

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + (V_0 + V_{eff})\psi, \tag{1}$$

where $V_0 \equiv V_0(\vec{r})$ is the confining potential and V_{eff} is the effective potential which takes into account the inter-atomic interactions inside the condensate, as determined in its simplest form by $V_{eff}(\vec{r},t) = g|\psi(\vec{r},t)|^2$, where g is a constant [3, 4].

Let us consider the situation where this wave equation can be replaced by a kinetic equation. In order to construct such an equation, we introduce the Wigner function associated with ψ, such that [5]

$$W(\vec{r},\vec{k},t) = \int \psi(\vec{r}+\vec{s}/2,t)\psi^*(\vec{r}-\vec{s}/2,t)\exp(-\vec{k}\cdot\vec{s})\, d\vec{s}. \tag{2}$$

It is then possible to derive (see the Appendix) the following evolution equation for the Wigner function

$$\left(\frac{\hbar^2}{2m}\vec{k}\cdot\nabla - i\hbar\frac{\partial}{\partial t}\right) W = -2V\sin\Lambda\, W, \tag{3}$$

where a bi-directional differential operator is denoted by

$$\Lambda = \leftarrow \left(\frac{\partial}{\partial\vec{r}}\cdot\frac{\partial}{\partial\vec{p}}\right) \rightarrow . \tag{4}$$

This operator acts to the left on V and to the right on W [5]. In this equation, the potential is

$$V = V_0 + g\int W(\vec{r},\vec{k},t)\frac{d\vec{k}}{(2\pi)^3} + \delta V, \tag{5}$$

where

$$\delta V = g\left(|\psi(\vec{r},t)|^2 - \int W(\vec{r},\vec{k},t)\frac{d\vec{k}}{(2\pi)^3}\right) \tag{6}$$

can be considered a noise term associated with the square mean deviations of the quasi probability, determined by the Wigner function W with respect to the local quantum probability, determined by the wave function ψ.

Equation (3) can be seen as the WME describing the space and time evolution of the BECs, and it is exactly equivalent to the GPE (1). However, it is of little use in the above exact form, and it is convenient to introduce some simplifying assumptions. This is justified for the important case of slowly varying potentials. In this case, we can neglect the higher order space derivatives, and introduce the approximation $\sin\Lambda \sim \Lambda$. This corresponds to the quasi-classical approximation, where the quantum potential fluctuations can also be neglected, viz. $\delta V \to 0$. Introducing these two simplifying assumptions, valid in the quasi-classical limit, we reduce the WME to a much simpler form

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla + \vec{F}\cdot\frac{\partial}{\partial\vec{k}}\right)W = 0, \tag{7}$$

where $\vec{v} = \hbar\vec{k}/m$ is the velocity of the condensate atoms corresponding to the wavevector state $\vec{k}$, and $\vec{F} = \nabla V$ is a force associated with the inhomogeneity of the condensate self-potential. The nonlinear term in the GPE (1) is hidden inside this force $\vec{F}$. As we will see, this nonlinear term will look very much like a ponderomotive force term, similar to the radiation pressure.

It should be noticed that this new equation is a closed kinetic equation for the Wigner function W. In this quasi-classical limit, W is just the particle occupation number for translational states with momentum $\vec{p} = \hbar\vec{k}$. Equation (7) is equivalent to a conservation equation, stating the conservation of the quasi-probability W in the six-dimensional classical phase space $(\vec{r},\vec{k})$, and can also be written as

$$\frac{d}{dt}W(\vec{r},\vec{k},t) = 0. \tag{8}$$

This kinetic equation can then be used to describe physical processes occurring in a BEC, as long as the quasi-classical approximation of slowly varying potentials is justified. The interest of such a kinetic descriptions will be illustrated with the aid of two simple and

different examples, to be presented in the next two sections. Many other applications can be envisaged, and will be explored in future publications.

III. SELF-PHASE MODULATION OF A BEAM CONDENSATE

Let us first consider the kinetic description of self-phase modulation of a BEC gas, moving with respect to the confining potential $V_0(\vec{r})$. Here, we can explore the analogy of this problem with that of self-phase modulation of short laser pulses moving in a nonlinear optical medium and well known in the literature [15]. In order to simplify our description, we consider the one-dimensional problem of a beam moving along the z-axis and neglect the axial variation of the background potential $\partial V_0/\partial z \sim 0$. The radial structure of the beam can easily be introduced later, and will not essentially modify the results obtained here. The kinetic equation (7) can then be written as

$$\left(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} + F_z \frac{\partial}{\partial k} \right) W(z, k, t) = 0, \tag{9}$$

with v_z and F_z given by, respectively,

$$v_z = \frac{\hbar k}{m} + g \frac{\partial}{\partial t} I(z, t) \quad , \quad F_z = \frac{dk}{dt} = -g \frac{\partial}{\partial z} I(z, t), \tag{10}$$

where we have used the intensity of the beam condensate, as defined by

$$I(z, t) = \int W(z, k, t) \frac{dk}{2\pi}. \tag{11}$$

Let us assume that the ultra-cold atomic beam has a mean velocity $v_0 = \hbar k_0/m$. This suggests the use of a new space coordinate $\eta = z - v_0 t$. In terms of this new coordinate, the quasi-classical equations of motion of a cold atom in the beam can be written as

$$\begin{aligned} \frac{d\eta}{dt} &= \frac{\partial h}{\partial k} = \frac{1}{m}(k - k_0), \\ \frac{dk}{dt} &= -\frac{\partial h}{\partial \eta} = -\frac{g}{\hbar} \frac{\partial}{\partial \eta} I(\eta, t), \end{aligned} \tag{12}$$

where we have introduced the Hamiltonian function

$$h(\eta, k, t) = \omega(\eta, k, t) - k v_0 = \frac{k}{m} \left(\frac{k}{2} - k_0 \right) + \frac{g}{\hbar} I(\eta, t). \tag{13}$$

Here $\omega(\eta, k, t)$ is the Hamiltonian in the rest frame expressed in the new coordinate. A straightforward integration of the equations of motion leads to

$$k(t) = k_0 - \frac{g}{\hbar} \int_0^t \frac{\partial}{\partial \eta} I(\eta, t') dt'. \tag{14}$$

At this point it is useful to introduce the concept of the beam energy chirp, $< \epsilon(\eta, t) >$, in analogy with the frequency chirp of short laser pulses [15]. By definition, it will be the beam mean energy at a given position and at a given time

$$< \epsilon(\eta, t) >= \hbar \int W(\eta, k, t) \omega(\eta, k, t) \frac{dk}{2\pi}, \tag{15}$$

where the weighting function $W(\eta, k, t)$ is the solution of the kinetic one-dimensional equation (9). A formal solution of this equation can be written as

$$W(\eta, k, t) = W(\eta_0(\eta, k, t), k_0(\eta, k, t), t_0), \tag{16}$$

where η_0 and k_0 are the initial conditions corresponding to the observed values at time t, as determined by the dynamical equations (13). Replacing it in equation (14), we obtain

$$< \epsilon(\eta, t) >= \hbar \int W(\eta_0, k_0, t_0) \left[\frac{k^2}{2m} + \frac{g}{\hbar} I(\eta, t) \right] \frac{dk}{2\pi}. \tag{17}$$

From equation (14) we notice that $dk = dk_0$. Neglecting higher order nonlinearities, we can then rewrite the above expression as

$$< \epsilon(\eta, t) >=< \epsilon(0) > - \frac{k_0}{m} g \int_0^t \frac{\partial}{\partial \eta} I(\eta, t') dt', \tag{18}$$

where $< \epsilon(0) > \equiv < \epsilon(\eta_0, t_0) >$ is the initial beam energy chirp.

Let us first consider that the beam profile $I(\eta)$ is independent of time. This is, of course, only valid for very short time intervals where the beam velocity dispersion is negligible. In this simple case, we have

$$< \epsilon(\eta, t) >=< \epsilon(0) > - \hbar v_0 g \left(\frac{\partial I}{\partial \eta} \right) t. \tag{19}$$

The maximum energy shift will be attained at some position inside the beam profile, $\eta = \eta_{max}$, determined by the stationarity condition

$$\frac{\partial}{\partial \eta} < \epsilon(\eta, t) >= \left(\frac{\partial^2 I}{\partial \eta^2}\right) = 0. \tag{20}$$

In order to deduce more specific answers, let us assume a Gaussian beam profile

$$I(\eta) = I_0 \exp\left(-\eta^2/\sigma^2\right), \tag{21}$$

where σ determines the beam width. For this profile we have $\eta_{max} = \pm\sigma/\sqrt{2}$, which leads to the following value of the maximum energy shift

$$\Delta\epsilon(t) \equiv < \epsilon(t) >_{max} - < \epsilon(0) >= \pm\frac{\hbar\sqrt{2}}{\sigma} g v_0 I_0 e^{-1/2} t. \tag{22}$$

This is similar to a well known result in nonlinear optics, stating that the maximum energy chirp due to a self-phase modulation is proportional to t, or to the distance travelled by the beam, $d = v_0 t$. This result clearly indicates that the initial beam will eventually split into two parts, one being accelerated to higher translational speeds, and the other being decelerated. This would correspond to the red-shift and to the blue-shift observed in nonlinear optics. The decelerated beam will eventually stop after a time $t \simeq \tau$, such that $\Delta\epsilon(\tau) =< \epsilon(0) >$. This will determine the condition for translational beam freezing.

It should be noticed that the same result could also be obtained directly from the GPE (1). But the interest of the present derivation is that it demonstrates the irrelevance of the phase of the wave function ψ, because such a phase was ignored in our kinetic calculation. Therefore, instead of the self-phase modulation, it would be more appropriate to call it a beam self-deceleration.

Another interesting aspect of our kinetic approach is that it can be easily refined, as shown briefly here. Let us then improve the above calculation by considering the beam dispersion. This will inevitably occur because of the linear velocity dispersion of the atomic beam. Such a dispersion will decrease the chirping effect, because of the decrease in time of $(\partial I/\partial \eta)$. In order to model it, we can assume a time-varying Gaussian beam shape, as described by

$$I(\eta, t) = I_0 \left(\frac{\sigma_0}{\sigma(t)}\right)^{1/2} \exp\left(-\frac{\eta^2}{\sigma(t)^2}\right). \tag{23}$$

If we now consider that $\sigma(t) = \sigma_0(1 + \delta t^2)$, where $\delta = (2m/\hbar^2)\Delta\epsilon_0/\sigma_0$ is proportional to the initial energy spread $\Delta\epsilon_0$. This will lead to a maximum energy shift

$$\Delta\epsilon_d(t) = \frac{\ln(t)}{t}\Delta\epsilon(t). \tag{24}$$

It is clear that the linear beam velocity dispersion will decrease the maximum attainable chirp, by changing the linearity with time into a logarithmic law. However, this will only occur for very long times, $t \sim 1/\sqrt{\delta}$, which for ultra-cold atomic beams with a very low translational energy dispersion $\Delta\epsilon_0$ will not be relevant.

The other cause of the beam dispersion is the nonlinear process itself, which will eventually break up the initial pulse into two distinct pulses. In this case, the self-phase modulation process will not be attenuated because the beam width will be conserved, but the two secondary pulses will suffer self-phase modulation themselves, and will eventually break up later, resulting in the formation of several secondary pulses with different mean energy. However, the nonlinear dispersion will also be negligible as far as $\sigma_0^2 > 4m|\Delta\epsilon(t)|t^2/\hbar^2$. A more complete description of all these dispersion regimes can be obtained by solving numerically the kinetic equation (9).

IV. KINETIC DESCRIPTION OF BOGOLIUBOV OSCILLATIONS

The second example of an application of the kinetic equation for the BECs deals with the dispersion relation of sound waves. For simplicity, we consider again the one-dimensional model and neglect the radial structure of the oscillations. This allows us to treat the lowest order oscillating modes of the condensate. We assume some given equilibrium distribution $W_0(z, k, t)$, corresponding, for instance, to the Thomas-Fermi equilibrium solution in a given confining potential $V_0(\vec{r}_\perp, z)$ [16], and after linearization of the one-dimensional kinetic equation (9) with respect to the perturbation $\tilde{W}$, we obtain

$$\left(\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z}\right)\tilde{W}(z,k,t) + \tilde{F}\frac{\partial}{\partial k}W_0(z,k,t) = 0, \tag{25}$$

where the perturbed force is determined by

$$\tilde{F} = -\frac{g}{\hbar}\frac{\partial}{\partial z}\tilde{I}(z,t) = -\frac{g}{\hbar}\frac{\partial}{\partial z}\int \tilde{W}(z,k,t)\frac{dk}{2\pi}. \tag{26}$$

Let us now assume perturbations of the form $\tilde{W}, \tilde{I} \sim \exp(ikz - i\omega t)$. From the above equations we then obtain a relation between the perturbation amplitude of the Wigner

function $\tilde{W}$ and the perturbed beam intensity $\tilde{I}$

$$\tilde{W} = -\frac{gk}{\hbar(\omega - kv')}\tilde{I}\frac{\partial}{\partial k'}W_0(k'), \tag{27}$$

where we now specify the particle wavenumber state with k', in order to avoid confusion with the wavenumber k of the oscillation that we intend to study. The velocity corresponding to this particle state is $v' = \hbar k'/m$. Integration over the momentum spectrum of the particle condensate will then lead to the following expression

$$1 + \frac{g}{\hbar}k\int \frac{\partial W_0(k')/\partial k'}{(\omega - \hbar k k'/m)}\frac{dk'}{2\pi} = 0. \tag{28}$$

This is the kinetic dispersion relation for axial perturbations in the BECs. Let us illustrate this result by considering a simple case for a condensate beam with no translational dispersion, or with a translational temperature exactly equal to zero. The equilibrium state of the beam can then be described by

$$W_0(k') = 2\pi n_0 \delta(k' - k'_0), \tag{29}$$

where $n_0 = \int W_0(k')dk'/2\pi$ is the particle number density in the condensate. Replacing this in the dispersion relation (28). we have

$$1 - \frac{gk^2}{m}\frac{n_0}{(\omega - kv'_0)^2} = 0, \tag{30}$$

where $v'_0 = \hbar k'_0/m = p'_0/m$ is the beam velocity. This can also be written as

$$(\omega - kv'_0)^2 = k^2 c_s^2, \tag{31}$$

where

$$c_s = \sqrt{gn_0/m} \tag{32}$$

is nothing but the Bogoliubov sound speed. Obviously, equation (31) is the Doppler shifted dispersion relation of sound waves in the BEC gas, In its reference frame it reduces to $\omega = kc_s$.

Let us now consider a situation where, instead of the distribution (29), we have a beam with a small translational velocity spread, such that the number of particles with a velocity

$v' \sim c_s$ is small but nonzero. In this case, the resonant contribution in the integral of equation (28) has to be retained, although it is still possible to neglect the kinetic corrections in the principal part of the integral. The dispersion relation can then be written, in the condensate frame of reference, as

$$1 - \frac{k^2 c_s^2}{\omega^2} - \frac{i}{2} \frac{gm}{\hbar^2} \left(\frac{\partial W_0}{\partial k'} \right)_{k'=k'_s} = 0, \tag{33}$$

where $k'_s = mc_s/\hbar$ is the resonant momentum. The imaginary term in this equation can then lead to damping of the sound waves. Writing now $\omega = kc_s + i\gamma$, with $|\gamma| \ll kc_s$, we can then obtain the expression for the damping coefficient

$$\gamma = \frac{\omega}{4} \frac{gm}{\hbar^2} \left(\frac{\partial W_0}{\partial k'} \right)_{k'=k'_s}. \tag{34}$$

The above expression corresponds to the non-collisional Landau damping of Bogoliubov oscillations in the BECs. The present approach can also be generalized in a straightforward way to higher order oscillations of the condensate, where the radial structure has to be taken into account [13, 17].

V. CONCLUSIONS

In this paper, we have proposed a kinetic view of the Bose-Einstein condensate physics, based on the Wigner-Moyal equation. In the quasi-classical limit, the latter can be reduced to a closed kinetic equation for the corresponding Wigner function. The kinetic approach to BEC can be seen as an intermediate step between the GPE and the hydrodynamical equations for the condensate gas, often found in the literature.

We have discussed two different physical problems, in order to illustrate the versatility of the kinetic theory. One is a self-phase modulation of a BEC beam. The other is the dispersion relation of the Bogoliubov oscillations in the condensate gas. The first example shows that, due to the influence of its own inhomogeneous self-potential, nearly half of the beam is accelerated while the other half is decelerated. Under certain conditions, the decelerated part of the beam will tend to a state of complete halt. The second example shows that a kinetic dispersion relation for sound waves in BECs can be established, where Landau damping is automatically included. The present results only considered the lowest

order modes, but the same approach can be used to describe higher order oscillations of the BECs, including their radial structures, as well as the coupling with a background thermal gas. This investigation is, however, beyond the scope of the present work.

Several other different problems relevant to BECs can also be considered in the frame of the kinetic theory, such as modulational instabilities [18] and the wakefield generation. This indicates that the kinetic theory is a very promising approach to the physics of BECs, which will eventually allow us to introduce new ideas in this stimulating area of research, and to suggest new configurations to the experimentalists. However, the present work also clearly states that the present theory is only valid in the quasi-classical limit and for that reason some relevant problems, where the phase of the BEC wave function plays an important role, can only be treated by means of the GPE. Surprisingly, the self-phase modulation is not one of them, as demonstrated here.

Acknowledgments

One of the authors (J.T.M.) appreciates the hospitality of the Rutherford Appleton Laboratory, and partial support of the Centre for Fundamental Physics.

APPENDIX: Derivation of the Wigner-Moyal equation

In the present derivation we follow a procedure already used in other cases, for instance, in the case of electromagnetic waves moving in a space and time dependent dielectric medium [19]. For a different but nearly equivalent derivation of the WME see the appendix of reference [8]. Let us consider two distinct sets of values for space and time coordinates, $(\vec{r}_1, t_1)$ and $(\vec{r}_2, t_2)$, and let us use the notation $\psi_j = \psi(\vec{r}_j, t_j)$ and $V_j = V(\vec{r}_j, t_j)$, for $j = 1, 2$. This allows us to write two versions of the GPE (1) as

$$\left(\frac{\hbar^2}{2m} \nabla_j^2 - i\hbar \frac{\partial}{\partial t_j} \right) \psi_j = -V_j \psi_j. \tag{35}$$

Multiplying the equation $j = 1$ by ψ_2^* and the conjugate of the equation $j = 2$ by ψ_1, and subtract the resulting equations, we obtain

$$\left[\frac{\hbar^2}{2m} (\nabla_1^2 - \nabla_2^2) - i\hbar \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \right] C_{12} = -(V_1 - V_2) C_{12}, \tag{36}$$

where we have used $C_{12} = \psi_1 \psi_2^*$. The above equation suggests the use of two pairs of space and time variables, such that

$$\begin{aligned} \vec{r}_1 = \vec{r} - \vec{s}/2 \quad &, \quad t_1 = t - \tau/2, \\ \vec{r}_2 = \vec{r} + \vec{s}/2 \quad &, \quad t_2 = t + \tau/2. \end{aligned} \tag{37}$$

We can then rewrite the above equation as

$$\left[\frac{\hbar^2}{m} \frac{\partial}{\partial \vec{r}} \cdot \frac{\partial}{\partial \vec{s}} - i\hbar \frac{\partial}{\partial t} \right] C_{12} = -(V_1 - V_2) C_{12}. \tag{38}$$

It can also easily be shown, by developing the potentials V_j around $V(\vec{r}, t)$, that

$$(V_1 - V_2) = 2 \sinh \left(\frac{\vec{s}}{2} \cdot \frac{\partial}{\partial \vec{r}} + \frac{\tau}{2} \frac{\partial}{\partial t} \right) V(\vec{r}, t). \tag{39}$$

Let us now introduce the double Fourier transformation of the function $C_{12} \equiv C(\vec{r}, \vec{s}, t, \tau)$ on the variables $\vec{s}$ and τ, as defined by

$$W(\vec{r}, t, \omega, \vec{k}) = \int d\vec{s} \int d\tau C(\vec{r}, \vec{s}, t, \tau) \exp(-i\vec{k} \cdot \vec{s} + i\omega\tau). \tag{40}$$

This can be rewritten in terms of the wave function ψ as

$$W(\vec{r},t,\omega,\vec{k}) = \int d\vec{s} \int d\tau \psi(\vec{r}+\vec{s}/2, t+\tau/2)\psi^*(\vec{r}-\vec{s}/2, t-\tau/2)\exp(-i\vec{k}\cdot\vec{s}+i\omega\tau). \quad (41)$$

Using such a development in equation (38), we obtain for the Fourier amplitudes

$$\left(\frac{\hbar^2}{m}\vec{k}\cdot\frac{\partial}{\partial\vec{r}} - \hbar\frac{\partial}{\partial t}\right) W = -2V\sin\Lambda' W, \quad (42)$$

where we have used the following differential operator, operating backwards on the potential $V(\vec{r},t)$ and forward on W, viz.

$$\Lambda' = \frac{1}{2}^{\leftarrow}\left(\frac{\partial}{\partial\vec{r}}\cdot\frac{\partial}{\partial\vec{k}} - \frac{\partial}{\partial t}\frac{\partial}{\partial\omega}\right)^{\rightarrow}. \quad (43)$$

This is a formidable equation for W, which can be simplified by noting that the GPE implies the existence of a well defined relation between energy and momentum. This means that ω must be equal to some function of $\vec{k}$, or $\omega = \omega(\vec{k})$. Hence, we can state that

$$W(\vec{r},t,\omega,\vec{k}) = W(\vec{r},\vec{k},t)\delta(\omega-\omega(\vec{k})). \quad (44)$$

This leads to a much simpler evolution expression for $W(\vec{r},\vec{k},t)$. Before writing it down, we should also notice that the nonlinear term in V depends on $|\psi|^2$, and not on the function W. Thus, we can finally write

$$\left(\frac{\hbar^2}{2m}\vec{k}\cdot\nabla - i\hbar\frac{\partial}{\partial t}\right) W = -2(V_0 + g|\psi|^2)(\sin\Lambda)W, \quad (45)$$

where Λ is a simpler differential operator defined by

$$\Lambda = \leftarrow\left(\frac{\partial}{\partial\vec{r}}\cdot\frac{\partial}{\partial\vec{p}}\right)\rightarrow. \quad (46)$$

The function $W(\vec{r},\vec{k},t)$ can be seen as the Wigner function associated with the GPE, and equation (45) as the WME equation that describes its spatio-temporal behavior. This equation is equivalent to the initial wave equation (1), but it is not a closed equation for the quasi-probability function W. Therefore, some simplifying assumptions have to be introduced in order to make it more tractable, as explained in Section 2.

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